A MIRROR SYMMETRIC SOLUTION TO THE QUANTUM TODA LATTICE

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Abstract. We give a representation-theoretic proof of a conjecture from [28] providing integral formulas for solutions to the quantum Toda lattice in general type. This result generalizes work of Givental for SL_n/B and can be interpreted as a kind of mirror theorem for the full flag variety G/B. We also prove the existence of a totally positive critical point of the 'superpotential' in every mirror fiber.

1. Introduction

In [28] we introduced a conjectural 'mirror datum' $(Z_P, \omega, \mathcal{F}_P)$ associated to a general flag variety G/P, and used it to recover the quantum cohomology ring $qH^*(G/P)_{(q)}$ with quantum parameters inverted in its presentation due to Dale Peterson.

The goal of this paper is to show, in the full flag variety case, that associated integrals

$$S_\Gamma(h) = \int_{[\Gamma_h]} e^{\mathcal{F}_B}\ \omega_h, \tag{1.1}$$

defined in terms of the mirror datum $(Z_B, \omega, \mathcal{F}_B)$ of G/B, and with integration cycles satisfying certain technical conditions, are annihilated by the quantum Toda Hamiltonian associated to $G^\vee$ [13]. By Kim [10], the quantum Toda equations of $G^\vee$ are the 'quantum differential equations' of G/B in the sense of Givental, whose symbols recover the relations in the quantum cohomology ring. See also [1]. Therefore this result is in a sense a 'quantization' of [28].

In type A integral solutions of the form (1.1) to the quantum Toda lattice were obtained earlier by Givental [6], in what he called a 'mirror theorem' for SL_n/B, using ingenious and very explicit coordinates. The general mirror family introduced in [28] was inspired by this construction and is such that one recovers Givental's mirror family by restricting to a certain open subset of Z_B and considering a special choice of coordinates there.

Givental's construction of solutions to the quantum Toda lattice was also recently revisited by Gerasimov, Kharchev, Lebedev and Oblezin [4] (GKLO), who gave a new proof of Givental's type A result using Kostant's Whittaker model. Their proof has some features in common with our construction [28] but still relied in an essential way on the use of Givental's special coordinates, and hence didn't readily generalize to other types.

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In this paper we use a combination of methods from [4] and [28] to prove that the integrals (1.1) provide solutions to the quantum Toda lattice in general type. We also describe explicitly two distinguished choices for the families of integration cycles. The one focused on in [4] is a non-compact one, which we think of in the general setting as running through the 'totally negative parts' of the mirror fibers. The other distinguished family consists of cycles that are compact.

From now on we swap the roles of G and its Langlands dual compared to [28], as the 'A-model' will not enter into the picture much anymore. So the goal is to construct solutions to the quantum Toda lattice for G.

A T-equivariant analogue of the main result of this work will be treated in a sequel paper.

2. Notation and Preliminaries

We refer to [29, 11] for background on algebraic groups and basic representation theory. Let G be a simple, simply connected algebraic group over $\mathbb{C}$ of rank n with split real form. We fix opposite Borel subgroups $B = B_-$ and B_+ with unipotent radicals U_- and U_+, respectively. Assume that B_+ and B_- are also defined over $\mathbb{R}$ and the maximal torus $T = B_+ \cap B_-$ is split. Let $W = N_G(T)/T$ denote the Weyl group.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote by $\mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_-, \mathfrak{u}_+, \mathfrak{h}$ the Lie algebras of B_-, B_+, U_-, U_+ and T, respectively. Let $\mathfrak{h}_\mathbb{R}$ be the real form of $\mathfrak{h}$ and $\mathfrak{g}_\mathbb{R}$ the real form of $\mathfrak{g}$.

We set $I = \{1, \dots, n\}$ and choose $\{\alpha_i \mid i \in I\}$ to be the set of simple roots associated to the positive Borel B_+. We may view the α_i as elements of $\mathfrak{h}^*$ or as characters on T. The coroot associated to α_i is denoted $\alpha_i^\vee$ and gives a 1-parameter subgroup of T denoted $s \mapsto s^{\alpha_i^\vee}$.

Corresponding to the positive and negative simple roots we have Chevalley generators $e_i, f_i \in \mathfrak{g}$ and one parameter subgroups,

$$x_i(t) := \exp(te_i), \qquad y_i(t) := \exp(tf_i), \qquad t \in \mathbb{C},$$

in U_+ and U_-, respectively. Let

$$\dot{s}_i = x_i(-1)y_i(1)x_i(-1). \tag{2.1}$$

This element represents a simple reflection s_i in the Weyl group W.

For $w \in W$, a representative $\dot{w} \in G$ is defined by $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_m}$, where $s_{i_1}s_{i_2}\cdots s_{i_m}$ is a (any) reduced expression for w. The length m of a reduced expression for w is denoted by $\ell(w)$.

Let $\langle\ ,\ \rangle$ be the W-invariant inner product on $\mathfrak{h}^*$ such that $\langle \alpha, \alpha \rangle = 2$ for any long root α. We also denote the corresponding inner product on $\mathfrak{h}$ in the same way, by $\langle\ ,\ \rangle$.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{Z}(\mathfrak{g})$ its center, and let

$$\gamma : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W,$$

be the Harish-Chandra homomorphism. Define $c_2 \in \mathcal{Z}(\mathfrak{g})$ by $\gamma(c_2)(a) = \langle a, a \rangle$ for $a \in \mathfrak{h}^*$.

For any integral dominant weight λ we have an irreducible representation $V(\lambda)$ of G. In each $V(\lambda)$ let us fix a highest weight vector v_λ^+. Then for any $v \in V(\lambda)$

and extremal weight vector $\dot{w} \cdot v_\lambda^+$ we have the coefficient $\langle v, \dot{w} \cdot v_\lambda^+ \rangle \in \mathbb{C}$ defined by

$$v = \langle v, \dot{w} \cdot v_\lambda^+ \rangle \, \dot{w} \cdot v_\lambda^+ + \text{other weight space summands.}$$

We define $v_\lambda^- := \dot{w}_0 \cdot v_\lambda^+$. The most important choices for λ are the fundamental weights ω_i, where $i \in I$, and $\rho := \sum_{i \in I} \omega_i$.

Let $\mathcal{B}$ be the set of Borel subgroups in G with the conjugation action,

$$g \cdot B := gBg^{-1} \qquad g \in G, \; B \in \mathcal{B}.$$

Then we may identify $\mathcal{B}$ with the flag variety G/B_- in the usual way, by sending gB_- to $g \cdot B_-$. We define the intersection of two opposed big Bruhat cells

$$\mathcal{R}_{1,w_0} := (B_+ B_- \cap B_- \dot{w}_0 B_-)/B_-,$$

which can also be written more symmetrically as

$$\mathcal{R}_{1,w_0} = B_+ \cdot B_- \cap B_- \cdot B_+.$$

$\mathcal{R}_{1,w_0}$ is an open subset of G/B_- whose complement is an anti-canonical divisor. Note also that any line bundle on G/B_- becomes trivial when restricted to $\mathcal{R}_{1,w_0}$, since $\mathcal{R}_{1,w_0}$ lies inside the big cell $B_+ B_-/B_- \cong \mathbb{C}^N$.

3. The Toda lattice

The classical Toda lattice is an integrable system with Hamiltonian

$$H(x_i, p_i) = \frac{1}{2} \sum p_i^2 + \sum e^{x_i - x_{i+1}},$$

which was solved by Moser [22] in the 1970's. This integrable system was generalized to arbitrary G by Kostant [15], using phase space $T^*(T_{\mathrm{ad}}) \hat{=} T_{\mathrm{ad}} \times \mathfrak{h}^*$, the cotangent bundle to the adjoint torus T_{ad}, and Hamiltonian

$$H(t, h^*) = \frac{1}{2} \langle h^*, h^* \rangle + \sum_i \alpha_i(t),$$

where the simple root α_i is understood as a character on T_{ad}. Kostant then solved this system using a carefully chosen embedding of the phase space into the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, given by

$$(t, h^*) \mapsto F + h^* + \sum_{i \in I} \alpha_i(t) {f_i}^*. \tag{3.1}$$

Here ${e_i}^*, {f_i}^* \in \mathfrak{g}^*$ are defined to take value 1 on e_i and f_i, respectively, and vanish on all other weight spaces of $\mathfrak{g}$. Also $F = \sum e_i^*$. Note that $\mathfrak{h}^*, \mathfrak{b}_-^*$ are subspaces of $\mathfrak{g}^*$. The image of the phase space is in fact the translate by F of a B_--coadjoint orbit in $\mathfrak{b}_-^*$. Moreover, the Toda Hamiltonian now appears naturally as the restriction of (essentially) the Killing form, and a full set of Poisson-commuting constants of motion comes from restricting the remaining generators of $\mathbb{C}[\mathfrak{g}^*]^G$.

3.1. **The (Givental-)Kim presentation of** $qH^*(G^\vee/B^\vee)$**.** In [10] Kim described the relations of the small quantum cohomology ring of $G^\vee/B^\vee$ in terms of constants of motion of the Toda lattice associated to G, generalizing his joint work with Givental [7] in type A. Formally, this goes as follows. Let $\mathcal{A}^*$ denote the image of the embedding (3.1),

$$\mathcal{A}^* = F + \mathfrak{h}^* + \sum(\mathbb{C} \setminus \{0\})\ f_i{}^*,$$

and let $\phi_1, \dots, \phi_n$ be a set of homogeneous generators for $\mathbb{C}[\mathfrak{h}^*]^W$, following Chevalley. Consider the 'diagonalization' map

$$\Sigma : \mathcal{A}^* \to \mathfrak{h}^*/W.$$

That is, Σ corresponds to

$$\mathbb{C}[\mathfrak{h}^*]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathcal{A}^*],$$

where the first map is given by Chevalley's restriction theorem, and the third map comes from the inclusion $\mathcal{A}^* \hookrightarrow \mathfrak{g}^*$. Then for the ring $qH^*(G^\vee/B^\vee)[q_1^{-1}, \dots, q_n^{-1}]$ with quantum parameters inverted, Kim's presentation takes the form of an isomorphism

$$qH^*(G^\vee/B^\vee, \mathbb{C})[q_1^{-1}, \dots, q_n^{-1}] \xrightarrow{\sim} \mathbb{C}\left[\mathcal{A}^* \underset{\mathfrak{h}^*/W}{\times} \{0\}\right] = \mathbb{C}[\mathcal{A}^*]/(\Sigma_1, \dots, \Sigma_n),$$

where the $\Sigma_i := \Sigma^*(\phi_i)$ are precisely Kostant's constants of motion for the Toda lattice. In this presentation the usual Chern class generators x_i of the quantum cohomology ring appear via the isomorphism $H^2(G^\vee/B^\vee) \cong (\mathfrak{h}^\vee)^*$ and the identification of $(\mathfrak{h}^\vee)^*$ with functions on the $\mathfrak{h}^*$-part of $\mathcal{A}^*$. Explicitly, the x_i and q_i are identified with the coordinates on the degenerate leaf in $\mathcal{A}^*$ defined by

$$h^* = -\sum_{i \in I}(x_1 + \ldots + x_i)\alpha_i, \qquad t = \prod_{i \in I}(-q_i)^{\omega_i^\vee},$$

where h^* and t are as in (3.1).

3.2. **Peterson's presentation of** $qH^*(G^\vee/B^\vee)$**.** The mirror symmetric approach to the quantum cohomology rings [28] is more closely related to an alternative presentation of $qH^*(G^\vee/B^\vee)$ due to Dale Peterson [23], see also [16, 17, 25, 27, 28]. Namely consider the closed subvariety Y in G/B_- defined by

$$Y = \left\{gB_- \mid (g^{-1} \cdot F)|_{[\mathfrak{u}_-, \mathfrak{u}_-]} = 0\right\},$$

which is called the Peterson variety. Then in Peterson's presentation the quantum cohomology ring appears as ring of regular functions on an open stratum of Y. Explicitly, in the case where we localize at the quantum parameters,

$$qH^*(G^\vee/B^\vee, \mathbb{C})[q_1^{-1}, \dots, q_n^{-1}] \cong \mathbb{C}\left[\ Y \underset{G/B_-}{\times} \mathcal{R}_{1,w_0}\right].$$

This presentation relates to Kim's presentation by the map taking $uB_- \mapsto u^{-1} \cdot F$ for $u \in U_+ \cap B_- \dot{w}_0 B_-$, which defines a morphism

$$Y \underset{G/B_-}{\times} \mathcal{R}_{1,w_0} \to \mathcal{A}^* \underset{\mathfrak{h}^*/W}{\times} \{0\}.$$

That this is an isomorphism follows from work of Kostant, see [12].

Peterson's presentation via his variety Y has some distinct advantages over Kim's presentation. Namely, the Peterson variety also sees the quantum cohomology rings

of partial flag varieties $G^\vee/P^\vee$, by intersection with associated smaller Bruhat cells, and it provides presentations over the integers in general type [23].

4. The Quantum Toda lattice

The quantum Toda Hamiltonian we are interested in is the differential operator on $\mathfrak{h}$ defined by

$$\mathcal{H}_G = \frac{1}{2}\Delta \ - \frac{1}{z^2}\sum_{i\in I} e^{\alpha_i}, \tag{4.1}$$

where Δ be the Laplace operator associated to the W-invariant inner product $\langle \ , \ \rangle$ on $\mathfrak{h}$, and the α_i are the simple roots inside $\mathfrak{h}^*$. This can be understood as a quantization via the orbit method (for the solvable group B_-), which gives rise to Kostant's 'Whittaker model' for the centre of the universal enveloping algebra inside $\mathcal{U}(\mathfrak{b}_-)$. The quantum Toda lattice was constructed, analysed and solved by Kostant in [13, 14].

As Kazhdan and Kostant observed [3], the operator $\mathcal{H}_G$ can be obtained from the Laplace operator on G itself by restricting to appropriate 'Whittaker functions'. This approach is the analogue of the embedding of $T^*(T_{\mathrm{ad}})$ into a 'tridiagonal' part of $\mathfrak{g}^*$ in order to realize the classical Toda Hamiltonian as coming from an invariant quadratic form. The higher Casimirs generalizing the Laplace operator provide quantum integrals of motion, in analogy with the Poisson commuting constants of motion obtained in the classical case as generators of the ring of invariants (the direct translation from quantum integrals to classical integrals of motion being given by the Harish-Chandra homomorphism).

In Section 4.1 we will review and adapt to our situation this construction of the quantum Toda lattice, as Whittaker functions will turn out to appear very naturally in the mirror model. The main goal of the present paper is to connect Kostant's representation theoretic approach to the quantum Toda lattice with the B-model of $G^\vee/B^\vee$, and use it to prove the mirror conjecture from [28] for full flag varieties.

4.1. **Whittaker functions and the quantum Toda Hamiltonian.** Let $\chi_+ : \mathfrak{u}_+ \to \mathbb{C}$ and $\chi_- : \mathfrak{u}_- \to \mathbb{C}$ be Lie algebra homomorphisms. Then χ_+ and χ_- are determined by $\chi_+(e_i) =: \chi_+^{(i)}$, and $\chi_-(f_i) =: \chi_-^{(i)}$, respectively. Namely,

$$\begin{aligned} \chi_+ &= \sum_{i\in I} \chi_+^{(i)} e_i^*, \\ \chi_- &= \sum_{i\in I} \chi_-^{(i)} f_i^*. \end{aligned}$$

By abuse of notation we denote by e_i^* also the map $U_+ \to \mathbb{C}$ which takes u to the coefficient of e_i in the series expansion $u = \exp(n) = 1 + \sum e_i^*(n)e_i + \dots$, and similarly for f_i^*. The holomorphic characters of U_+ and U_-, respectively, corresponding to χ_+ and χ_- are denoted by

$$\begin{aligned} e^{\chi_+} :& \quad U_+ \to \mathbb{C}^*, \\ e^{\chi_-} :& \quad U_- \to \mathbb{C}^*. \end{aligned}$$

If for all i the coefficient $\chi_+^{(i)} \neq 0$, then e^{χ_+} and also χ_+ are called non-degenerate. Analogously for χ_-.

For our purposes the 'Whittaker functions' will not be functions on G but functions on the universal cover of the open dense subset, U_+TU_- of G. That is, they are defined on

$$X := U_+TU_- \underset{T}{\times} \mathfrak{h}.$$

More generally we will be interested in subsets of X of the form

$$X_{\mathcal{O}} := U_+TU_- \underset{T}{\times} \mathcal{O},$$

where $\mathcal{O}$ is a connected open subset in $\mathfrak{h}$ or $\mathfrak{h}_{\mathbb{R}}$. We call a smooth function $f : X_{\mathcal{O}} \to \mathbb{C}$ a *Whittaker function* with respect to χ_+ and χ_- if

$$f(u_+e^h u_-, h) = e^{\chi_+(u_+)} f(e^h, h) e^{\chi_-(u_-)}. \tag{4.2}$$

Clearly f on $X_{\mathcal{O}}$ is completely determined by its restriction to the Cartan component, and conversely any smooth function on $\mathcal{O}$ gives rise to a Whittaker function on $X_{\mathcal{O}}$.

Consider the representation of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(X_{\mathcal{O}})$ defined by

$$\xi \cdot f(u_+e^h u_-, h) := \left.\frac{d}{ds}\right|_{s=0} f(\exp(-s\xi)u_+e^h u_-, h_s), \qquad \xi \in \mathfrak{g}_{\mathbb{R}},$$

where $s \mapsto h_s$ is the lift of the torus factor of $\exp(-s\xi)u_+e^h u_-$ to $\mathfrak{h}_{\mathbb{R}}$ with $h_0 = h$, defined for small enough s.

The connection between the quantum Toda Hamiltonian and Whittaker functions can now be stated as follows, compare [3].

(1) If f is a Whittaker function (for χ_+ and χ_-) and c is a central element in $\mathcal{U}(\mathfrak{g})$, then $c \cdot f$ is again a Whittaker function. The resulting action of $\mathcal{Z}(\mathfrak{g})$ on Whittaker functions on $X_{\mathcal{O}}$ factors through a representation of $\mathcal{Z}(\mathfrak{g})$ on $C^\infty(\mathcal{O})$, by restriction to the Cartan factor and extension.

(2) Let c_2 be the degree 2 Casimir such that $\gamma(c_2)(a) = \langle a, a\rangle$. Then the action of c_2 on $C^\infty(\mathcal{O})$ as defined in (1) is given by

$$c_2 \cdot f = e^\rho \left(\frac{1}{2}\Delta + \sum_{i\in I} \chi_+^{(i)}\chi_-^{(i)} e^{\alpha_i}\right)(e^{-\rho} f), \qquad f \in C^\infty(\mathcal{O}).$$

Constructing solutions to the quantum Toda lattice is thereby equivalent to constructing Whittaker functions annihilated by generators $\gamma^{-1}(\phi_i)$ of $\mathcal{Z}(\mathfrak{g})$. [Explicitly, we will use $\chi_+^{(i)} = -\frac{1}{z}$ and $\chi_-^{(i)} = \frac{1}{z}$, to find solutions for (4.1).] The construction of appropriate Whittaker functions is done via Kostant's theory of 'Whittaker modules' and their (Whittaker vector) matrix coefficients [13, 14].

4.2. **The A-model of $G^\vee/B^\vee$.** The quantum Toda lattice for G appears in the A-model of $G^\vee/B^\vee$ as the natural quantization of the (Givental-)Kim presentation for the quantum cohomology. In that context it is sometimes called the *quantum cohomology D-module*, or the *quantum differential equations* of $G^\vee/B^\vee$, see [5, 10]. Officially this means that it consists of the differential operators annihilating the components of Givental's J-function,

$$J(t_1, \dots, t_n) \in H^*(G^\vee/B^\vee, \mathbb{C}[t_1, \dots, t_n, z^{-1}])[[e^{t_1}, \dots, e^{t_n}]],$$

which is defined in terms of descendent 2-point Gromov-Witten invariants. Therefore the Gromov-Witten theory of $G^\vee/B^\vee$ provides a full set of solutions inside

$\mathbb{C}[t_1, \dots, t_n, z^{-1}][[e^{t_1}, \dots, e^{t_n}]]$ to the quantum Toda lattice for G, namely by expansion of the J-function with respect to the Schubert basis.

A direct connection between Givental's J-function, albeit in the G-equivariant case, and Whittaker modules (inside completed Verma modules) was made by A. Braverman [1].

4.3. **The B-model of $G^\vee/B^\vee$.** A conjectural B-model to $G^\vee/B^\vee$ was constructed in [28]. In this B-model there is a family (parameterized by $\mathfrak{h}$) of holomorphic volume forms on $\mathcal{R}_{1,w_0}$, giving rise to 'period integrals' varying over the family. The quantum Toda equations are supposed to describe the variation of these periods as $h \in \mathfrak{h}$ varies.

For SL_n the mirrors, written down in explicit coordinates by Givental [6], can be embedded into $\mathcal{R}_{1,w_0}$ in such a way that his B-model reappears as a restriction to an open subset of our construction from [28]. The connection between Givental's mirrors and Kostant's Whittaker modules in type A was worked out by GKLO [4].

Details of the general construction of a B-model for $G^\vee/B^\vee$ follow in Section 6. In preparation we recall the definition of our basic holomorphic N-form on $\mathcal{R}_{1,w_0}$ and a particular variant of it coming from GKLO.

5. Two regular N-forms on $\mathcal{R}_{1,w_0}$

The following is a special case of Proposition 7.2 in [28].

Proposition 5.1. *Let $\mathbf{i} = (i_1, \dots, i_N) \in I^N$ correspond to a reduced expression $s_{i_1} s_{i_2} \dots s_{i_N}$ of w_0 in W. There is a unique holomorphic N-form $\omega_{\mathbf{i}}$ on $\mathcal{R}_{1,w_0}$ such that the restriction of $\omega_{\mathbf{i}}$ to the open subset*

$$\mathcal{R}_{\mathbf{i}} = \{x_{i_1}(a_1) \cdots x_{i_N}(a_N)B_- \mid a_i \in \mathbb{C}^*\}$$

in $\mathcal{R}_{1,w_0}$ is given by

$$\frac{da_1}{a_1} \wedge \frac{da_2}{a_2} \wedge \dots \wedge \frac{da_N}{a_N}.$$

If $\mathbf{j}$ is another reduced expression of w_0, and is related to $\mathbf{i}$ by a single braid relation of length m, then

$$\omega_{\mathbf{j}} = (-1)^{m+1} \omega_{\mathbf{i}}.$$

In particular the form $\omega_{\mathbf{i}}$ is independent of the reduced expression $\mathbf{i}$ up to sign. □

We note that the subset $\mathcal{R}_{\mathbf{i}}$ in $\mathcal{R}_{1,w_0}$ is the open stratum in a stratification introduced by Deodhar [2, 19].

In the following we assume a reduced expression $\mathbf{i}$ has been chosen, and suppress the subscript $\mathbf{i}$, referring to the N-form defined in Proposition 5.1 simply as ω. The variant of our form ω defined by

$$\omega_{GKLO}(uB_-) := \left\langle u \cdot v_\rho^-, v_\rho^+ \right\rangle \omega(uB_-), \tag{5.1}$$

is a Lie-theoretic version of the N-form introduced by [4] in the type A case.

Proposition 5.2. *For any $g \in G$ denote by $\kappa_g : G/B_- \to G/B_-$ the map of left translation, $\kappa_g(g'B_-) = gg'B_-$. Let ω and ω_{GKLO} be viewed as rational N-forms*

on G/B_-. Then we have the identities

$$\kappa_g^*\omega(uB_-) = \frac{\left\langle u\cdot v_\rho^-, v_\rho^+\right\rangle}{\left\langle gu\cdot v_\rho^-, v_\rho^+\right\rangle \left\langle gu\cdot v_\rho^-, v_\rho^-\right\rangle}\omega(uB_-), \tag{5.2}$$

$$\kappa_g^*\omega_{GKLO}(uB_-) = \frac{1}{\left\langle gu\cdot v_\rho^-, v_\rho^-\right\rangle^2}\omega_{GKLO}(uB_-). \tag{5.3}$$

Remark 5.3. Proposition 5.2 implies the following identities.

(1) For $h \in \mathfrak{h}$ and Chevalley generators f_i, e_i the volume form ω transforms according to

$$\begin{aligned}
\kappa_{\exp(h)}^*\omega &= \omega,\\
\kappa_{y_i(s)}^*\omega &= \left(1+e_i^*(u)s\right)^{-1}\omega,\\
\kappa_{x_i(s)}^*\omega &= \left(1+\frac{\left\langle u\cdot v_\rho^-, f_i\cdot v_\rho^+\right\rangle}{\left\langle u\cdot v_\rho^-, v_\rho^+\right\rangle}s\right)^{-1}\omega.
\end{aligned}$$

(2) The alternative volume form ω_{GKLO} is U_+-invariant. In particular it is well-defined on the entire big cell U_+B_-/B_-. However it is not T-invariant, satisfying instead

$$\kappa_{\exp(h)}^*\omega_{GKLO} = e^{2\rho(h)}\omega_{GKLO}.$$

Proof. It is easy to see that the formulas (5.2) and (5.3) are equivalent to one another. Now suppose $g_1, g_2 \in G$ are such that the identity (5.3) holds. We claim that

$$\kappa_{g_2}^*\left(\kappa_{g_1}^*\omega_{GKLO}\right)(uB_-) = \frac{1}{\left\langle g_1g_2u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\omega_{GKLO}(uB_-). \tag{5.4}$$

If so, then the formula (5.3) follows also for $g = g_1g_2$, and therefore we need to check it (or equivalently (5.2)) only on a generating subset of G.

To prove the claim compute

$$\begin{aligned}
\kappa_{g_2}^*\left(\kappa_{g_1}^*\omega_{GKLO}\right)(uB^-) &= \kappa_{g_2}^*\left(\frac{1}{\left\langle g_1u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\omega_{GKLO}\right)(uB_-) = \\
&\kappa_{g_2}^*\left(uB_-\mapsto\frac{1}{\left\langle g_1u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\right)\frac{1}{\left\langle g_2u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\omega_{GKLO}(uB_-).
\end{aligned} \tag{5.5}$$

To apply $\kappa_{g_2}^*$ above we need a factorization

$$g_2u = u_{g_2}b_{g_2}$$

where $u_{g_2}\in U_+$ and $b_{g_2}\in B_-$. Then

$$\begin{aligned}
\kappa_{g_2}^*\left(uB_-\mapsto\frac{1}{\left\langle g_1u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\right)(uB_-) &= \frac{1}{\left\langle g_1u_{g_2}\cdot v_\rho^-, v_\rho^-\right\rangle^2}\\
= \frac{1}{\left\langle g_1g_2ub_{g_2}^{-1}\cdot v_\rho^-, v_\rho^-\right\rangle^2} &= \frac{1}{\left\langle g_1g_2u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\frac{1}{\rho(b_{g_2})^2}\\
&= \frac{1}{\left\langle g_1g_2u\cdot v_\rho^-, v_\rho^-\right\rangle^2}\left\langle g_2u\cdot v_\rho^-, v_\rho^-\right\rangle^2.
\end{aligned} \tag{5.6}$$

The claim now follows from the combination of (5.5) and (5.6).

To finish the proof it suffices to check the identities from Remark 5.3. Let us choose a reduced expression $\mathbf{i}$ of w_0 such that ω restricted to

$$\mathcal{R}_{\mathbf{i}} = \{x_{i_1}(a_1)\dots x_{i_N}(a_N)B_- \mid a_i \neq 0\}$$

takes the form

$$\frac{da_1}{a_1} \wedge \dots \wedge \frac{da_N}{a_N}.$$

Note that it is clear from the definition that ω is invariant under translation by elements of T. In fact $\mathcal{R}_{\mathbf{i}}$ is itself a much bigger torus, $\mathcal{R}_{\mathbf{i}} \cong (\mathbb{C}^*)^N$, on which ω is the standard invariant N-form. We are therefore left with two kinds of transformations to consider, $\kappa_{y_i(s)}$ and $\kappa_{x_i(s)}$.

(1) To work out the coordinate transformation corresponding to $\kappa_{y_i(s)}$ we note that $y_i(s)x_j(a) = x_j(a)y_i(s)$ for $i \neq j$ and

$$y_i(s)x_i(a) = x_i\left(\frac{a}{1+as}\right)\left(\frac{1}{1+as}\right)^{\alpha_i^\vee} y_i\left(\frac{s}{1+as}\right).$$

Supose $1 \le l_1 < \dots < l_m \le N$ are the indices for which $i_{l_j} = i$. Applying the above identities repeatedly we obtain

$$y_i(s)x_{i_1}(a_1)\dots x_{i_N}(a_N) = x_{i_1}(a_1)\dots x_{i_{l_1}}(a'_{l_1})\left(\frac{a'_{l_1}}{a_{l_1}}\right)^{\alpha_i^\vee} x_{i_{l_1+1}}(a_{l_1+1})\dots$$
$$\dots x_{i_{l_2}}(a'_{l_2})\left(\frac{a'_{l_2}}{a_{l_2}}\right)^{\alpha_i^\vee}\dots x_{i_{l_m}}(a'_{l_m})\left(\frac{a'_{l_m}}{a_{l_m}}\right)^{\alpha_i^\vee}\dots x_{i_N}(a_N)y_i\left(\frac{s}{1+(a_{l_1}+\dots+a_{l_m})s}\right).$$

where

$$a'_{l_j} = \frac{a_{l_j}(1+(a_{l_1}+\dots+a_{l_{j-1}})s)}{1+(a_{l_1}+\dots+a_{l_j})s}.$$

Since ω restricted to $\mathcal{R}_{\mathbf{i}}$ is invariant under the action of the 'big' torus, that is $\mathcal{R}_{\mathbf{i}}$ itself, we may disregard the factors $\left(\frac{a'_{l_j}}{a_{l_j}}\right)^{\alpha_i^\vee}$. Thus ω transforms under $\kappa_{y_i(s)}$ as under the coordinate transformation

$$(a_1,\dots,a_N) \mapsto (a'_1,\dots,a'_N),$$

where $a'_j := a_j$ if $j \notin \{l_1,\dots,l_m\}$. Since this coordinate transformation is lower triangular its Jacobian is easily computed to be

$$\det\left(\frac{\partial a'_j}{\partial a_k}\right)_{j,k} = \prod_{j=1}^m\left(\frac{1+(a_{l_1}+\dots+a_{l_{j-1}})s}{1+(a_{l_1}+\dots+a_{l_j})s}\right)^2 = \left(\frac{1}{1+(a_{l_1}+\dots+a_{l_m})s}\right)^2.$$

Note also the telescopic product identity

$$\frac{1}{\prod_{j=1}^N a'_j} = (1+(a_{l_1}+\dots+a_{l_m})s)\ \frac{1}{\prod_{j=1}^N a_j}.$$

Therefore we obtain

$$\frac{da'_1}{a'_1}\wedge\frac{da'_2}{a'_2}\wedge\cdots\wedge\frac{da'_N}{a'_N}=\frac{1}{\prod_{j=1}^N a'_j}\left(\frac{1}{1+(a_{l_1}+\ldots+a_{l_m})s}\right)^2 da_1\wedge da_2\wedge\cdots\wedge da_N$$
$$=\frac{1}{1+(a_{l_1}+\ldots+a_{l_m})s}\omega.$$

Clearly $a_{l_1}+\ldots+a_{l_m}$ is nothing other than $e_i^*(u)$ for $u=x_{i_1}(a_1)\ldots x_{i_N}(a_N)$, confirming the identity from Remark 5.3.

(2) To apply $\kappa^*_{x_i(s)}$ to ω let us assume without loss of generality that the reduced expression $\mathbf{i}$ of w_0 begins with $i_1=i$. Then $\kappa_{x_i(s)}$ corresponds to the coordinate transformation

$$(a_1,a_2,\ldots,a_N)\mapsto(a_1+s,a_2,\ldots,a_N),$$

and therefore

$$\kappa^*_{x_i(s)}\omega=\frac{a_1}{a_1+s}\omega.$$

It is easy to see that for $u=x_{i_1}(a_1)\ldots x_{i_N}(a_N)$ we have indeed

$$\left(1+\frac{\langle u\cdot v_\rho^-,f_i\cdot v_\rho^+\rangle}{\langle u\cdot v_\rho^-,v_\rho^+\rangle}s\right)^{-1}=\left(1+\frac{1}{a_1}s\right)^{-1}=\frac{a_1}{a_1+s}.$$

□

6. The mirror family to $G^\vee/B^\vee$

In this section we will review the ingredients of mirror symmetry for the full flag variety $G^\vee/B^\vee$ following [28]. We will consider from the start the variant of the mirror family over $\mathfrak{h}$, rather than the family over T. Also, for convenience, we have chosen G simply connected, while in [28] the group on the mirror symmetric side was adjoint. The mirror family over $\mathfrak{h}$ is however unaffected by this change. Finally, we replace the mirror family from [28] by its translate by $\dot w_0$, which will turn out to be more natural in connection with the Whittaker functions we are constructing.

6.1. Let

$$Z:=\{(g,h)\in U_+TU_-\underset{T}{\times}\mathfrak{h}\mid g\cdot B_+=B_-\}. \tag{6.1}$$

Z is viewed as a family of varieties via the map $\mathrm{pr}_2: Z\to\mathfrak{h}$ projecting onto the second factor. For $h\in\mathfrak{h}$ let us write

$$Z_h:=U_+e^hU_-\cap B_-\dot w_0=\{g\in U_+e^hU_-\mid g\cdot B_+=B_-\}.$$

Z_h may be identified with the fiber $\mathrm{pr}_2^{-1}(h)$ in Z. We record the following basic properties of the family Z.

(1) Fix $h\in\mathfrak{h}$. Then the fiber Z_h is isomorphic to the intersection of opposite big cells, $\mathcal{R}_{1,w_0}$, via the map

$$\begin{array}{rccc}\beta_h: & Z_h & \longrightarrow & \mathcal{R}_{1,w_0}\\ & ue^h\bar u^{-1} & \mapsto & u^{-1}\cdot B_-=e^h\bar u^{-1}\cdot B_+.\end{array}$$

In particular Z_h is smooth of dimension $N=\dim_{\mathbb{C}}(G/B)$.

(2) The isomorphisms from (1) can be combined to give a trivialization

$$\beta : Z \xrightarrow{\sim} \mathcal{R}_{1,w_0} \times \mathfrak{h}, \tag{6.2}$$

where $(ue^h\bar{u}^{-1}, h) \mapsto (u^{-1} \cdot B_-, h)$.

Note that in particular

$$Z_0 = U_+U_- \cap B_-\dot{w}_0 = \{u\bar{u}^{-1} \in U_+U_- \mid u^{-1} \cdot B_- = \bar{u}^{-1} \cdot B_+\},$$

and we have the isomorphism

$$\beta_0 : Z_0 \xrightarrow{\sim} \mathcal{R}_{1,w_0}, \qquad u\bar{u}^{-1} \;\mapsto\; u^{-1} \cdot B_- = \bar{u}^{-1} \cdot B_+. \tag{6.3}$$

6.2. We define a function $\mathcal{F}$ on Z (the 'superpotential') as follows.

$$\mathcal{F}(ue^h\bar{u}^{-1}, h; z) = \frac{1}{z}\left(\sum_{i \in I} e_i^*(u) + \sum_{i \in I} f_i^*(\bar{u})\right), \tag{6.4}$$

where we may think of z as a positive parameter $z \in \mathbb{R}_{>0}$. The restriction of $\mathcal{F}$ to a fiber Z_h of the mirror family (or rather $Z_h \times \mathbb{R}_{>0}$) is denoted by $\mathcal{F}_h$.

Remark 6.1. Note that for fixed z the function $e^{\mathcal{F}(\ ;z)} : Z \to \mathbb{C}$ is the restriction to Z of the 'trivial' Whittaker function,

$$\begin{array}{rcl} \mathcal{W}_0 : U_+TU_- \underset{T}{\times} \mathfrak{h} & \to & \mathbb{C}, \\ (u_+e^hu_-, h) & \mapsto & e^{\chi_+(u_+)}e^{\chi_-(u_-)}, \end{array}$$

corresponding to characters χ_+, χ_- defined by $\chi_+(e_i) = \frac{1}{z}$ and $\chi_-(f_i) = -\frac{1}{z}$, see also Section 4.1.

6.3. Let $\mathbf{i}$ be a reduced expression for w_0 and $\omega_{\mathbf{i}}$ the N-form on $\mathcal{R}_{1,w_0}$ defined in Proposition 5.1. Denote by ω_Z or $\omega_{\mathbf{i},Z}$ the pullback of $\omega_{\mathbf{i}}$ to Z along the map

$$\begin{array}{rccc} pr_1 \circ \beta : & Z & \to & \mathcal{R}_{1,w_0}, \\ & (ue^h\bar{u}^{-1}, h) & \mapsto & u^{-1} \cdot B_-, \end{array}$$

and write ω_h or $\omega_{\mathbf{i},h}$ for the pullback of ω_Z to the fiber Z_h.

Note that the non-canonical choice of a reduced expression $\mathbf{i}$ will affect at most the sign of ω_Z. In some special cases we will consider later on, this sign will cancel out against a sign coming from the choice of orientation for the integration cycle, making for a canonical solution to the quantum Toda lattice.

The mirror datum to $G^\vee/B^\vee$ is now made up of the three ingredients introduced above: the family $Z \to \mathfrak{h}$, the regular function $\mathcal{F}$, and the holomorphic N-form ω_Z on Z. We may denote it compactly as $(Z, \omega_Z, \mathcal{F})$.

6.4. **A translation action on Z.** We can transfer the natural action of the additive group of $\mathfrak{h}$ on the product $\mathfrak{h} \times \mathcal{R}_{1,w_0}$, given by $h \cdot (h', g \cdot B_-) := (h' + h, g \cdot B_-)$, to Z using the trivialization β from (6.2).

Lemma 6.2. *Let $h \in \mathfrak{h}$ and $(h', g) \in Z$. The translation action of h on (h', g) takes the form*

$$h \cdot (h', g) = (h + h', ge^h).$$

Proof. We have that $g \in Z_{h'}$, hence we may write $g = ue^{h'}\bar{u}^{-1}$ for $u \in U_+, \bar{u} \in U_-$. Then

$$ge^h = ue^{h'}\bar{u}^{-1}\ e^h = ue^{h+h'}(\bar{u}')^{-1},$$

where $\bar{u}' = e^{-h}\bar{u}e^h \in U_-$. Also $ge^h \cdot B_+ = g \cdot B_+ = B_-$. Hence $ge^h \in Z_{h+h'}$. Moreover $\beta(h+h', ge^h) = (h+h', u^{-1} \cdot B_-)$ as required. □

7. Distinguished cycles for integration

7.1. The compact cycles $\Gamma^{(1)}$.

Definition 7.1. Let $h \in \mathfrak{h}$ and let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced expression of w_0. We may define $\Gamma^{(1)}_{\mathbf{i},h} \subset Z_h$ to be

$$\{ue^h\bar{u}^{-1} \in Z_h \mid u^{-1} \cdot B_- = x_{i_1}(a_1)\cdots x_{i_N}(a_N) \cdot B_- \text{ where } \|a_j\| = 1 \text{ for all } j\ \}.$$

Note that $\Gamma^{(1)}_{\mathbf{i},h}$ is naturally isomorphic to a compact torus $(S^1)^N$. We define an associated N-cycle $[\Gamma^{(1)}_{\mathbf{i},h}] \in H_N(Z_h, \mathbb{Z})$ by choosing the anti-clockwise orientation on each S^1 factor. The resulting family of integration contours is denoted $\Gamma^{(1)}_{\mathbf{i}} := ([\Gamma^{(1)}_{\mathbf{i},h}])_{h\in\mathfrak{h}}$, or just $\Gamma^{(1)}$ if the choice of reduced expression is clear or irrelevant.

Lemma 7.2. *The cycle $[\Gamma^{(1)}_{\mathbf{i},h}]$ defines a nonzero element of $H_N(Z_h, \mathbb{Z})$. Moreover, if two reduced expressions $\mathbf{i}$ and $\mathbf{j}$ are related by a braid relation of length m then*

$$[\Gamma^{(1)}_{\mathbf{i},h}] = (-1)^{m+1}[\Gamma^{(1)}_{\mathbf{j},h}].$$

In particular, the cycle $[\Gamma^{(1)}_{\mathbf{i},h}]$ is independent of the reduced expression $\mathbf{i}$ up to sign.

Proof. We may identify Z_h with $\mathcal{R}_{1,w_0}$ by β_h, and $\Gamma^{(1)}_{\mathbf{i},h}$ with

$$\Gamma_{\mathbf{i}} := \{x_{i_1}(a_1)\cdots x_{i_N}(a_N) \cdot B_- \mid \|a_j\| = 1 \text{ for all } j\ \},$$

and work in $\mathcal{R}_{1,w_0}$. By Section 5 we have a well defined N-form $\omega_{\mathbf{i}}$ on $\mathcal{R}_{1,w_0}$. It follows from

$$\int_{[\Gamma_{\mathbf{i}}]} \omega_{\mathbf{i}} = (2\pi i)^N$$

that $[\Gamma_{\mathbf{i}}]$ defines a nontrivial homology class in $H_N(\mathcal{R}_{1,w_0}, \mathbb{Z})$.

Now $\Gamma_{\mathbf{i}}$ lies in the open coordinate patch $\mathcal{R}_{\mathbf{i}}$ (see Proposition 5.1) while $\Gamma_{\mathbf{j}}$ lies inside $\mathcal{R}_{\mathbf{j}}$. Note that $\mathcal{R}_{\mathbf{i}}$ is isomorphic to $(\mathbb{C}^*)^N$ and therefore $[\Gamma_{\mathbf{i}}]$ generates $H_N(\mathcal{R}_{\mathbf{i}}, \mathbb{Z})$. Rescaling the S^1 factors in $[\Gamma_{\mathbf{j}}]$ we may replace this cycle by one that is homologous but lies in the intersection $\mathcal{R}_{\mathbf{i}} \cap \mathcal{R}_{\mathbf{j}}$, and therefore also defines a cycle $[\Gamma'_{\mathbf{j}}]$ in $H_N(\mathcal{R}_{\mathbf{i}}, \mathbb{Z})$, It follows that $[\Gamma'_{\mathbf{j}}]$ must be a multiple of the (generating) cycle $[\Gamma_{\mathbf{i}}]$ in $H_N(\mathcal{R}_{\mathbf{i}}, \mathbb{Z})$. Hence $[\Gamma_{\mathbf{j}}]$ is a multiple of $[\Gamma_{\mathbf{i}}]$ in $H_N(\mathcal{R}_{1,w_0}, \mathbb{Z})$. However we have by Proposition 5.1 that

$$\omega_{\mathbf{i}} = (-1)^{m+1}\omega_{\mathbf{j}}.$$

Therefore

$$\int_{[\Gamma_{\mathbf{j}}]} \omega_{\mathbf{i}} = (-1)^{m+1}\int_{[\Gamma_{\mathbf{j}}]} \omega_{\mathbf{j}} = (-1)^{m+1}(2\pi i)^N,$$

which implies $[\Gamma_{\mathbf{j}}] = (-1)^{m+1}[\Gamma_{\mathbf{i}}]$. □

7.2. **Total positivity and the non-compact integration cycles $\Gamma^{(w_0)}$.**

Definition 7.3 (Lusztig [18]). Inside the real points of a (split) real algebraic group G and each of its related varieties $X = T, U_+, U_-$ and G/B_-, there is a semi-algebraic subset $X^{>0}$ called the totally positive part, which is defined as follows.

The totally positive part of T is the precisely the subset of T for which all characters take values in $\mathbb{R}_{>0}$. Equivalently, if we consider the real points of T (isomorphic to $(\mathbb{R}^*)^n$), then $T^{>0}$ is the connected component of the identity.

For U_+ and U_- the totally positive parts are given by

$$\begin{aligned} U_+^{>0} &:= \{x_{i_1}(a_1)\dots x_{i_N}(a_N) \mid a_i \in \mathbb{R}_{>0}\}, \\ U_-^{>0} &:= \{y_{i_1}(a_1)\dots y_{i_N}(a_N) \mid a_i \in \mathbb{R}_{>0}\}, \end{aligned}$$

where $\mathbf{i} = (i_1, \dots, i_N)$ is a (any) reduced expression of w_0. And one puts these together to build

$$G^{>0} \quad := \quad U_+^{>0} T_{>0} U_-^{>0} = U_-^{>0} T_{>0} U_+^{>0},$$

where a proof of the last identity may be found in [18].

The totally positive part of the flag variety $\mathcal{B}$ is

$$\mathcal{B}^{>0} := U_+^{>0} \cdot B_- = U_-^{>0} \cdot B_+.$$

Again the last identity is proved in [18]. Note that the totally positive part of $\mathcal{B}$ lies in $\mathcal{R}_{1,w_0}$. We may also denote it by $\mathcal{R}_{1,w_0}^{>0}$.

These definitions generalize the classical notion of total positivity inside GL_n developed by Polya and Schoenberg among others. The introduction of a theory of total positivity to flag varieties is due to Lusztig, even in type A.

7.2.1. Let $h \in \mathfrak{h}_\mathbb{R}$, and note that $\exp(\mathfrak{h}_\mathbb{R}) = T_{>0}$. We can now use the trivialization β to pull back the totally positive part in $\mathcal{R}_{1,w_0}$ to the mirror fibers Z_h. We will refrain from calling this the totally positive part of Z_h, which instead will be defined differently in Section 11. Let

$$\begin{aligned} \Gamma_h^{(w_0)} := \{g \in Z_h \mid \beta_h(g) \in \mathcal{B}^{>0}\} = \{ue^h\bar{u}^{-1} \in Z_h \mid u^{-1} \in U_+^{>0}\ \} \\ = \{ue^h\bar{u}^{-1} \in Z_h \mid \bar{u}^{-1} \in U_-^{>0}\ \}, \end{aligned}$$

where the final equality uses that $e^h \in T_{>0}$. We will see in Section 11 that $\mathcal{F}_h$ always has a critical point in $\Gamma_h^{(w_0)}$.

For any choice of reduced expression $\mathbf{i}$ of w_0 we obtain a parameterization,

$$\begin{aligned} \mathbb{R}_{>0}^N &\xrightarrow{\sim} \Gamma_h^{(w_0)} \\ (a_1, \dots, a_N) &\mapsto \beta_h^{-1}(x_{i_1}(a_1)\dots x_{i_N}(a_N) \cdot B_-), \end{aligned}$$

which gives rise to an orientation on $\Gamma_h^{(w_0)}$. We denote by $[\Gamma_{\mathbf{i},h}^{(w_0)}]$ the oriented real (semi-algebraic) manifold inside Z_h obtained in this way, and by $\Gamma^{(w_0)} = \Gamma_{\mathbf{i}}^{(w_0)} = ([\Gamma_{\mathbf{i},h}^{(w_0)}])_{h\in\mathfrak{h}_\mathbb{R}}$, the corresponding family over $\mathfrak{h}_\mathbb{R}$.

By Proposition 5.1, if $\mathbf{j}$ is a reduced expression obtained from $\mathbf{i}$ by a braid relation of length m, then the corresponding (rational subtraction-free) coordinate transformation $(a_1, \dots, a_N) \mapsto (a_1', \dots, a_N')$ reverses orientations precisely if m is even. Therefore the orientation of $[\Gamma_{\mathbf{i},h}^{(w_0)}]$ depends on $\mathbf{i}$ in the same way.

7.3. **Conditions on more general families.** Let $\mathcal{O}$ be a connected open subset of $\mathfrak{h}_{\mathbb{R}}$ or $\mathfrak{h}$. Let $\Gamma = ([\Gamma_h])_{h\in\mathcal{O}}$ be a continuous family of real, possibly non-compact, N-dimensional semi-algebraic cycles $[\Gamma_h]$ in Z_h, for which $\mathrm{Re}(\mathcal{F}_h) \to -\infty$ in any non-compact direction of Γ_h, for a/any fixed $z > 0$. In this case

$$\int_{[\Gamma_h]} e^{\mathcal{F}} \omega_h \tag{7.1}$$

is well-defined and absolutely convergent, therefore differentiable in h.

We will also assume that $0 \in \mathcal{O}$, as we want $[\Gamma_h]$ to be equivalent to the translate $[h \cdot \Gamma_0]$ defined using Section 6.4, for small h. We remark that the formal setting for the cycles $[\Gamma_h]$ should be the N-th rapid decay homology group associated to an irregular rank one connection, $\nabla(f) = df - f\ d\mathcal{F}_h(\ ;z)$, defined by $\mathcal{F}_h$ on $\mathcal{R}_{1,w_0}$. This homology group was defined by Bloch and Esnault in dimension 1 and more recently generalized by Hien [8] to arbitrary dimension using work of T. Mochizuki [21]. The case of exponential connections is also treated in the earlier work of Hien and Roucairol [9].

We note that in certain cases we can start with a cycle $[\Gamma_0]$ in Z_0 and extend it automatically to a family over all of $\mathfrak{h}_{\mathbb{R}}$ with the above decay properties. Namely, suppose $[\Gamma_0]$ has the property that in any non-compact direction of Γ_0 we have $\mathrm{Re}(\mathcal{F}_0) \to -\infty$, and all individual summands $\mathrm{Re}(f_i^*(\bar{u}))$ and $\mathrm{Re}(e_i^*(u))$ are bounded from above. This means that no summand of $\mathrm{Re}(\mathcal{F}_0)$ can tend to $+\infty$. In that case we can use the translation action to define

$$\Gamma_h := h \cdot \Gamma_0 = \{u\bar{u}^{-1}e^h \in Z_h \mid u\bar{u}^{-1} \in \Gamma_0\},$$

for $h \in \mathfrak{h}_{\mathbb{R}}$. The claim that $[\Gamma_h]$ again has the same decay behavior for $\mathrm{Re}(\mathcal{F}_h)$ as $[\Gamma_0]$ for $\mathrm{Re}(\mathcal{F}_0)$ follows from the simple observation that

$$\mathcal{F}_h(u\bar{u}^{-1}e^h) = \frac{1}{z}\left(\sum_{i\in I} e_i^*(u) + \sum_{i\in I} f_i^*(e^{-h}\bar{u}e^h)\right) = \frac{1}{z}\left(\sum_{i\in I} e_i^*(u) + \sum_{i\in I} e^{\alpha_i(h)} f_i^*(\bar{u})\right),$$

and that $e^{\alpha_i(h)} > 0$ (since $h \in \mathfrak{h}_{\mathbb{R}}$).

Remark 7.4. The families of cycles $\Gamma^{(1)}$ and $\Gamma^{(w_0)}$ are obtained from $[\Gamma_0^{(1)}]$ and $[\Gamma_0^{(w_0)}]$ in this way.

For the proof of Proposition 10.3 we require another technical condition on Γ_0 (identified with its image in G/B_- under β_0). Namely for small $s > 0$ we assume that for the intersection with $\mathcal{R}_{1,w_0}$ of the closures of $\exp(sf_i)\cdot\Gamma_0$ and $\exp(se_i)\cdot\Gamma_0$, respectively, the real part of $\mathcal{F}_0$ tends to $-\infty$ in any unbounded direction. This is satisfied for the two specific families introduced above. The e_i and f_i could also be replaced by another set of generators of $\mathfrak{g}$, to make the condition weaker.

Another approach to constructing integration cycles, due to Givental, is recalled in Section 11.

8. Statement of the main theorem

Let $\mathcal{O}$ be a connected open subset either of $\mathfrak{h}_{\mathbb{R}}$ or of $\mathfrak{h}$ that contains 0. Let $\Gamma = ([\Gamma_h])_{h\in\mathcal{O}}$ be a family of real, possibly non-compact, N-dimensional semi-algebraic cycles in Z_h, as described in Section 7.3.

We fix a reduced expression $\mathbf{i}$ of w_0 and let $\omega_h = \omega_{\mathbf{i},h}$ be the N-form on Z_h defined in Section 6.3. Then let

$$S_\Gamma(h,z) := \int_{[\Gamma_h]} e^{\mathcal{F}}\ \omega_h, \tag{8.1}$$

for $h \in \mathcal{O}$ and $z \in \mathbb{R}_{>0}$. For example

$$S_{\Gamma^{(1)}}(h,z) = \int_{[\Gamma^{(1)}_{\mathbf{i},h}]} e^{\mathcal{F}}\ \omega_{\mathbf{i},h} \quad \text{and} \quad S_{\Gamma^{(w_0)}}(h,z) = \int_{[\Gamma^{(w_0)}_{\mathbf{i},h}]} e^{\mathcal{F}}\ \omega_{\mathbf{i},h}, \tag{8.2}$$

using the integration cycles defined in Section 7.

In general, the sign of $S_\Gamma(h,z)$ depends on the reduced expression $\mathbf{i}$ used to define ω_h. However the special solutions $S_{\Gamma^{(1)}}$ and $S_{\Gamma^{(w_0)}}$, where we have chosen the orientation of the integration cycle concurrently, are independent of $\mathbf{i}$. The integration cycles $\Gamma^{(w_0)}$ are the ones focused on in type A by GKLO [4]. Note also that $S_{\Gamma^{(1)}}(h,z)$ extends to a global holomorphic function on $\mathfrak{h}_{\mathbb{C}} \times \mathbb{C}^*$.

The following result was conjectured in [28].

Theorem 8.1. *The integrals* (8.1) *are solutions to the quantum Toda lattice. In particular they are annihilated by the quantum Toda Hamiltonian* (4.1)*.*

9. A $\mathcal{U}(\mathfrak{g})$-module structure on $\mathcal{H}ol(Z_0)$.

We consider the restriction of the complex line bundle $L_{-\rho} = G \times_{B_-} \mathbb{C}_{-\rho}$ to the intersection of opposite big cells $\mathcal{R}_{1,w_0}$. Since $\mathcal{R}_{1,w_0}$ is open in G/B_-, the representation of G on the space of sections induces a representation of $\mathfrak{g}$ on $\Gamma_{hol}(L_{-\rho}|_{\mathcal{R}_{1,w_0}})$, the space of holomorphic sections of the restricted line bundle. Moreover, since $\mathcal{R}_{1,w_0}$ is preserved by T, we have a corresponding representation of T on $\Gamma_{hol}(L_{-\rho}|_{\mathcal{R}_{1,w_0}})$ which is compatible with the $\mathfrak{g}$-module structure. Explicitly, let us set

$$\begin{aligned} &M_{-\rho} := \Gamma_{hol}(L_{-\rho}|_{\mathcal{R}_{1,w_0}}) \\ &= \{\tilde f\ : (U_+ \cap B_-\dot w_0 B_-)B_- \to \mathbb{C} \mid \tilde f \text{ homolorphic }, \tilde f(gb) = \tilde f(g)\rho(b),\ \forall b \in B_-\}. \end{aligned}$$

The actions of $\mathfrak{g}$ and T on $M_{-\rho}$ are given by

$$(X \cdot \tilde f)(g) \ := \ \left.\frac{d}{ds}\right|_{s=0} \tilde f(\exp(-sX)g), \tag{9.1}$$

$$t \cdot \tilde f(g) \ := \ \tilde f(t^{-1}g), \tag{9.2}$$

for $X \in \mathfrak{g}$ and $t \in T$. Note that the $\mathcal{U}(\mathfrak{g})$-module $M_{-\rho}$ has zero infinitesimal character, see [14, Proposition 5.1].

The restriction of $\tilde f \in M_{-\rho}$ to $U_+ \cap B_-\dot w_0 B_-$ defines an isomorphism,

$$\begin{aligned} M_{-\rho} \ &\xrightarrow{\sim} \ \mathcal{H}ol(\mathcal{R}_{1,w_0}), \\ \tilde f \ &\mapsto \ \Big(f : u \cdot B_- \mapsto \tilde f(u)\Big), \qquad \text{where } \ u \in U_+ \cap B_-\dot w_0 B_-, \end{aligned}$$

identifying $M_{-\rho}$ with holomorphic functions on $\mathcal{R}_{1,w_0}$. The actions of $\mathfrak{g}$ and T on $M_{-\rho}$ thereby carry over to representations on $\mathcal{H}ol(\mathcal{R}_{1,w_0})$.

Consider now the zero fiber, Z_0, of our mirror family. We obtain a $\mathfrak{g}$-module and compatible T-module structure on $\mathcal{H}ol(Z_0)$ via the isomorphism $\beta_0 : Z_0 \longrightarrow \mathcal{R}_{1,w_0}$ from (6.3). By construction, this representation of $\mathfrak{g}$ on $\mathcal{H}ol(Z_0)$ extends to a representation of $\mathcal{U}(\mathfrak{g})$ with zero infinitesimal character.

9.1. **A $\mathfrak{u}_+$-Whittaker vector.** Let $\chi : \mathfrak{u}_+ \to \mathbb{C}$ be the 1-dimensional representation defined by $\chi(e_i) = \frac{1}{z}$, for all $i \in I$, and consider the corresponding holomorphic character e^χ on U_+. Let $\psi_+ \in \mathcal{H}ol(Z_0)$ be defined by

$$\psi_+(u\bar{u}^{-1}) := e^{\chi(u)},$$

and let $\widetilde{\psi}_+ \in M_{-\rho}$ denote the section of $L_{-\rho}|_{\mathcal{R}_{1,w_0}}$ associated to ψ_+.

Lemma 9.1. *ψ_+ is a $\mathfrak{u}_+$-Whittaker vector in $\mathcal{H}ol(Z_0)$ with character χ. That is,*

$$e_i \cdot \psi_+ = \frac{1}{z}\psi_+, \qquad \text{for all } i \in I.$$

Proof. From the definitions we see

$$\begin{aligned}(e_j \cdot \psi_+)(u\bar{u}^{-1}) &= (e_j \cdot \widetilde{\psi}_+)(u^{-1}) = \left.\frac{d}{ds}\right|_{s=0} \widetilde{\psi}_+(\exp(-se_j)u^{-1}) \\ &= \left.\frac{d}{ds}\right|_{s=0} \exp\left(\frac{1}{z}\sum_{i\in I} e_i^*(u\exp(se_j))\right) = \left.\frac{d}{ds}\right|_{s=0} \exp\left(\frac{1}{z}\left(\sum_{i\in I} e_i^*(u) + s\right)\right) \\ &= \frac{1}{z}\psi_+(u\bar{u}^{-1}).\end{aligned}$$

□

Definition 9.2 ([14])**.** A $\mathcal{U}(\mathfrak{g})$-module with a cyclic Whittaker vector is called a *Whittaker module.*

Definition 9.3. Let V_+ be the $\mathcal{U}(\mathfrak{g})$-submodule of $\mathcal{H}ol(Z_0)$ generated by ψ_+.

By [14, Theorem 3.6.2] V_+ is an irreducible Whittaker module, see also [20]. Note that V_+ no longer has an action of T.

9.2. **A $\mathfrak{u}_-$-Whittaker vector.** Let $\bar{\chi} : \mathfrak{u}_- \to \mathbb{C}$ be the 1-dimensional representation defined by $\bar{\chi}(f_i) = \frac{1}{z}$, for all $i \in I$, and consider the corresponding holomorphic character $e^{\bar{\chi}}$ on U_-. Let $\psi_- \in \mathcal{H}ol(Z_0)$ be defined by

$$\psi_-(u\bar{u}^{-1}) := \frac{1}{\left\langle u^{-1} \cdot v_\rho^-, v_\rho^+ \right\rangle} e^{\bar{\chi}(\bar{u})}.$$

The functions ψ_+ and ψ_- are Lie-theoretic analogues of the functions introduced in terms of Givental coordinates in [4].

Lemma 9.4. *$\psi_- \in \mathcal{H}ol(Z_0)$ is a $\mathfrak{u}_-$-Whittaker vector with character $\bar{\chi}$, That is,*

$$f_i \cdot \psi_- = \frac{1}{z}\psi_-.$$

In the following lemma we collect some identities used in the proof of the Lemma 9.4.

Lemma 9.5. *Suppose $u \in U_+, \bar{u} \in U_-$ are given such that $u\bar{u}^{-1}$ lies in Z_0. Consider a fixed Chevalley generator f_i.*

(1) *For $s \in \mathbb{C}$ such that $1 + se_i^*(u) \neq 0$ we have the identity*

$$uy_i(s) = b_{(s)}u_{(s)},$$

where $b_{(s)} \in B_-$ and $u_{(s)} \in U_+$ are given explicitly by

$$\begin{aligned} b_{(s)} &= (1+se_i^*(u))^{\alpha_i^\vee}\, y_i\big(s(1+se_i^*(u))\big) \\ u_{(s)} &= y_i\big(-s(1+se_i^*(u))\big)\left(\frac{1}{1+se_i^*(u)}\right)^{\alpha_i^\vee} u\, y_i(s). \end{aligned}$$

(2) *Let s be as above, and define $\bar{u}_{(s)}$ by*

$$\bar{u}_{(s)}^{-1}\cdot B^+ = u_{(s)}^{-1}\cdot B^-.$$

Then $\bar{u}_{(s)} = \bar{u}\ y_i(s)$. The element $u_{(s)}\bar{u}_{(s)}^{-1}$ in Z_0 is given by

$$y_i\big(-s(1+se_i^*(u))\big)\left(\frac{1}{1+se_i^*(u)}\right)^{\alpha_i^\vee} u\bar{u}^{-1}.$$

The proof of the Lemma is straightforward.

Proof of Lemma 9.4. To analyze the action of f_i on ψ_- we use Lemma 9.5 with all its notations. Let $\widetilde{\psi}_-$ denote the element of $M_{-\rho}$ associated to ψ_-.

$$(f_i\cdot\psi_-)(u\bar{u}^{-1}) = (f_i\cdot\widetilde{\psi}_-)(u^{-1}) = \frac{d}{ds}\Big|_{s=0}\widetilde{\psi}_-(\exp(-sf_i)u^{-1})$$

$$= \frac{d}{ds}\Big|_{s=0}\widetilde{\psi}_-(u_{(s)}^{-1}b_{(s)}^{-1}) = \frac{d}{ds}\Big|_{s=0}\widetilde{\psi}_-(u_{(s)}^{-1})\rho(b_{(s)})^{-1} = \frac{d}{ds}\Big|_{s=0}\widetilde{\psi}_-(u_{(s)}^{-1})\frac{1}{(1+se_i^*(u))}.$$

Now

$$\widetilde{\psi}_-(u_{(s)}^{-1}) = \psi_-(u_{(s)}\bar{u}_{(s)}^{-1}) = \frac{1}{\left\langle u_{(s)}^{-1}\cdot v_\rho^-, v_\rho^+\right\rangle}e^{\bar{\chi}(\bar{u}_{(s)})} = \frac{(1+se_i^*(u))}{\left\langle u^{-1}\cdot v_\rho^-, v_\rho^+\right\rangle}e^{\bar{\chi}(\bar{u})+\frac{1}{z}s},$$

using the formulas for $u_{(s)}$ and $\bar{u}_s$ from Lemma 9.5. Therefore we get

$$(f_i\cdot\psi_-)(u\bar{u}^{-1}) = \frac{d}{ds}\Big|_{s=0}\frac{1}{\left\langle u^{-1}\cdot v_\rho^-, v_\rho^+\right\rangle}e^{\bar{\chi}(\bar{u})+\frac{1}{z}s} = \frac{1}{z}\psi_-(u\bar{u}^{-1}).$$

□

Definition 9.6. Let V_- be the $\mathcal{U}(\mathfrak{g})$ module in $\mathcal{H}ol(Z_0)$ generated by ψ_-. This is another irreducible Whittaker module.

10. A Whittaker functional on V_+

Consider again the holomorphic character $e^{\bar{\chi}}$ on U_- given by

$$\bar{u}\mapsto e^{\bar{\chi}(\bar{u})} = e^{\frac{1}{z}\sum f_i^*(\bar{u})}.$$

Suppose we have fixed a cycle $[\Gamma_0]$ in Z_0 as in Section 7.3. Then we define a linear map $\Psi_-^{[\Gamma_0]} : V_+ \to \mathbb{C}$ by

$$\Psi_-^{[\Gamma_0]}(f) := \int_{[\Gamma_0]} e^{\bar{\chi}(\bar{u})}\, f\ \omega.$$

Remark 10.1. Recall how the integration cycles were chosen in Section 7.3 for $e^{\mathcal{F}}$ to have exponential decay in any non-compact direction. Now $\Psi_-^{[\Gamma_0]}(\psi_+) = \int_{[\Gamma_0]} e^{\bar{\chi}(\bar{u})}\psi_+\ \omega = \int_{[\Gamma_0]} e^{\mathcal{F}}\ \omega$. Since repeated actions by elements of $\mathfrak{g}$ on the ψ_+ produce only rational amplitude factors which do not affect convergence, $\Psi_-^{[\Gamma_0]}$ is defined on all of V_+.

Let us consider $\Psi_-^{[\Gamma_0]}$ as an element of the $\mathcal{U}(\mathfrak{g})$-module V_+^* dual to V_+.

Proposition 10.2. *For all $i \in I$, we have*

$$f_i \cdot \Psi_- = \frac{1}{z}\Psi_-.$$

That is, Ψ_- is a $\mathfrak{u}_-$-Whittaker vector in V_+^*.

10.1. **A bilinear pairing.** To prove Proposition 10.2 we will construct a pairing between the Whittaker modules V_+ and V_-, such that the Whittaker vector ψ_- becomes identified with the Whittaker functional Ψ_- from Theorem 10.2. This pairing essentially generalizes one introduced in [4].

Let $[\Gamma_0]$ be a middle-dimensional cycle in Z_0 as above. Consider integrals of the form

$$\int_{[\Gamma_0]} \phi\ \psi\ \ \omega_{GKLO},$$

where ϕ, ψ are holomorphic functions on Z_0 and ω_{GKLO} is the volume form from (5.1), pulled back to Z_0 via the isomorphism $\beta_0 : Z_0 \to \mathcal{R}_{1,w_0}$.

Proposition 10.3. *Let $\phi \in V_-$ and $\psi \in V_+$.*

(1) *The formula*

$$\langle \phi, \psi \rangle_{[\Gamma_0]} := \int_{[\Gamma_0]} \phi\ \psi\ \omega_{GKLO}$$

defines a bilinear pairing, $\langle \phi, \psi \rangle_{[\Gamma_0]} : V_- \times V_+ \to \mathbb{C}$.

(2) *For any $X \in \mathfrak{g}$,*

$$\langle X \cdot \phi, \psi \rangle_{[\Gamma_0]} + \langle \phi, X \cdot \psi \rangle_{[\Gamma_0]} = 0.$$

Proof. The pairing $\langle\ ,\ \rangle_{[\Gamma_0]}$ is well-defined for the same reason as $\Psi_-^{[\Gamma_0]}$, see Remark 10.1. Let us now prove (2). We transfer the integrals from Z_0 to $\mathcal{R}_{1,w_0}$ via β_0, but keeping the notation the same. For a function f on $\mathcal{R}_{1,w_0}$ identified with a function on $x \in U_+ \cap B_-\dot{w}_0 B_-$, we denote by $\tilde{f}$ the corresponding element of $M_{-\rho}$ given by

$$\tilde{f}(xb_-) = f(x)\rho(b_-), \quad \text{for } b_- \in B_-,$$

and by $\bar{f}$ the usual function

$$\bar{f}(xb_-) = f(x), \quad \text{for } b_- \in B_-$$

on $(U_+ \cap B_-\dot{w}_0 B_-)B_-$ given by f.

The proposition relies on the observation that ω_{GKLO} 'compensates' for the weight $-\rho$ twists coming from the representation on $M_{-\rho}$. Namely, on the level of N-forms we claim that

$$(\exp(sX) \cdot \phi)(\exp(sX) \cdot \psi)\ \omega_{GKLO} = \kappa^*_{\exp(-sX)}(\phi\ \psi\ \omega_{GKLO}), \tag{10.1}$$

where on the left hand side we have the local action of $\exp(sX)$ on sections of $L_{-\rho}$, and on the right hand side we have the pull-back of forms, as in Proposition 5.2.

To prove (10.1), write $\exp(-sX)x = x_{(s)}b_-$ for $x_{(s)} \in U_+$ and $b_- \in B_-$. Then

$$(\exp(sX) \cdot \phi)(\exp(sX) \cdot \psi) = \tilde{\phi}(x_{(s)}b_-)\tilde{\psi}(x_{(s)}b_-) = \phi(x_{(s)})\psi(x_{(s)})\rho(b_-)^2.$$

On the other hand we have

$$\begin{aligned}\kappa^*_{\exp(-sX)}(\phi\ \psi\ \omega_{GKLO})(x) &= \bar\phi(\exp(-sX)x)\bar\psi(\exp(-sX)x)\kappa^*_{\exp(-sX)}\omega_{GKLO}\\ &= \bar\phi(\exp(-sX)x)\bar\psi(\exp(-sX)x)\frac{1}{\langle\exp(-sX)x\cdot v_{-\rho}, v_{-\rho}\rangle^2}\omega_{GKLO}\\ &= \phi(x_{(s)})\psi(x_{(s)})\frac{1}{\langle x_{(s)}b_-\cdot v_{-\rho}, v_{-\rho}\rangle^2}\omega_{GKLO} = \phi(x_{(s)})\psi(x_{(s)})\rho(b_-)^2\omega_{GKLO},\end{aligned}$$

using Proposition 5.2 and that $x_{(s)}\in U_+$. This completes the proof of the identity (10.1).

If Γ_0 is compact, then $\phi\ \psi\ \omega_{GKLO}$ being holomorphic implies that the left hand side of

$$\int_{[\Gamma_0]}\kappa^*_{\exp(-sX)}(\phi\ \psi\ \omega_{GKLO}) = \int_{[\Gamma_0]}(\exp(sX)\cdot\phi)(\exp(sX)\cdot\psi)\ \omega_{GKLO} \tag{10.2}$$

is constant for s small. Therefore the differential $\frac{d}{ds}\big|_0$ vanishes, implying (2) in this case,

$$\begin{aligned}&\frac{d}{ds}\Big|_0\int_{[\Gamma_0]}(\exp(sX)\cdot\phi)(\exp(sX)\cdot\psi)\ \omega_{GKLO}\\ &\qquad\qquad = \int_{[\Gamma_0]}(X\cdot\phi)\ \psi\ \omega_{GKLO} + \int_{[\Gamma_0]}\phi\ (X\cdot\psi)\ \ \omega_{GKLO} = 0.\end{aligned}$$

For more general $[\Gamma_0]$ as in Section 7.3 we consider X to be a generator, such as e_i or f_i, so that (10.2) converges for small $s>0$. Then we can use the exponential decay condition and an argument as in [9, Section 2] for the 'limit Stokes formula', to see that the differential of (10.2) vanishes. □

Proof of Proposition 9.4. For any $\psi\in V_+$ we have

$$f_i\cdot\Psi_-^{[\Gamma_0]}(\psi) = \langle\psi_-, -f_i\cdot\psi\rangle_{[\Gamma_0]} = \langle f_i\cdot\psi_-,\psi\rangle_{[\Gamma_0]} = \frac{1}{z}\langle\psi_-,\psi\rangle_{[\Gamma_0]} = \frac{1}{z}\Psi_-^{[\Gamma_0]}(\psi),$$

by Proposition 10.3 and then Lemma 9.4. □

Proof of Theorem 8.1. Consider the 'matrix coefficient'

$$g\mapsto\Psi_-^{\Gamma}(g^{-1}\cdot\psi_+). \tag{10.3}$$

Strictly speaking only $\mathcal{U}(\mathfrak{g})$ acts on V_+, so the above definition doesn't make sense for general $g\in G$. However, for $[\Gamma_0]$ a member of a family of cycles $[\Gamma_h]_{h\in\mathcal{O}}$ as in Section 7.3, this particular matrix coefficient turns out to be well defined on the corresponding $X_{\mathcal{O}}$ (compare Section 4.1), giving

$$\begin{aligned}X_{\mathcal{O}} := U_+TU_-\underset{T}{\times}\mathcal{O}\quad&\to\quad\mathbb{C}\\ (u_+e^hu_-,h)\quad&\mapsto\quad\Psi_-^{\Gamma}(u_-^{-1}e^{-h}u_+^{-1}\cdot\psi_+) = e^{\chi_-(u_-)}\Psi_-^{\Gamma}(e^{-h}\cdot\psi_+)e^{\chi_+(u_+)},\end{aligned}$$

where $\chi_+^{(i)} = -\frac{1}{z}$, $\chi_-^{(i)}=\frac{1}{z}$, for all i, and we compute $\Psi_-^{\Gamma}(e^{-h}\cdot\psi_+)$ below. In this way (10.3) gives rise to a Whittaker function, which we denote by $\mathcal{W}_\Gamma$.

Since V_+, and with it V_+^*, have zero infinitesimal character, $\mathcal{W}_\Gamma$ is annihilated by the Casimir generators in $\mathcal{Z}(\mathfrak{g})$. By Section 4.1 (2), therefore,

$$e^{-\rho}\ \mathcal{W}_\Gamma|_{\mathcal{O}}\quad:\quad h\mapsto e^{-\rho(h)}\ \Psi_-^{\Gamma}(e^{-h}\cdot\psi_+)$$

is a solution to the quantum Toda lattice (4.1).

To compute this solution, suppose $h \in \mathcal{O}$. Then

$$e^{-\rho(h)}\ \Psi^{\Gamma}_{-}(e^{-h}\cdot\psi_+) = e^{-\rho(h)}\int_{[\Gamma_0]} e^{\bar\chi(\bar u)}(e^{-h}\cdot\psi_+)(u\bar u^{-1})\omega_0 \tag{10.4}$$
$$= e^{-\rho(h)}\int_{[\Gamma_0]} e^{\bar\chi(\bar u)}(e^{-h}\cdot\widetilde{\psi}_+)(u^{-1})\omega_0 = e^{-\rho(h)}\int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ \widetilde{\psi}_+(e^{h}u^{-1})\omega_0$$
$$= e^{-\rho(h)}\int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ \widetilde{\psi}_+(e^{h}u^{-1}e^{-h})\rho(e^h)\omega_0 = \int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ \widetilde{\psi}_+(e^{h}u^{-1}e^{-h})\omega_0$$
$$= \int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ \psi_+(e^{h}ue^{-h})\omega_0 = \int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ e^{\chi(e^{h}ue^{-h})}\omega_0.$$

Now we can use Proposition 10.3 (2) to rewrite the integral

$$\int_{[\Gamma_0]} e^{\bar\chi(\bar u)}\ e^{\chi(e^{h}ue^{-h})}\omega_0 = \int_{[\Gamma_0]} e^{\chi(u)}\ e^{\bar\chi(e^{-h}\bar u e^{h})}\omega_0,$$

to get precisely

$$e^{-\rho(h)}\ \Psi_{-}(e^{-h}\cdot\psi_+) = \int_{[\Gamma_h]} e^{\chi(u)+\bar\chi(\bar u)}\ \omega_h = \int_{[\Gamma_h]} e^{\frac{1}{z}(\sum e_i^*(u)+\sum f_i^*(\bar u))}\ \omega_h, \tag{10.5}$$

where $[\Gamma_h]$ is the translate of $[\Gamma_0]$, as in Section 7.3. This completes the proof of the theorem. □

11. Total positivity and critical points of $\mathcal{F}_h$

Let us fix $z > 0$. By the mirror symmetric construction of the quantum cohomology ring of $G^\vee/B^\vee$ proved in [28], the critical points of the $\mathcal{F}_h = \mathcal{F}_h(\ ;z)$ (for varying h) sweep out the Peterson variety

$$Y_B^* = Y \underset{G/B}{\times} \mathcal{R}_{1,w_0},$$

with h determining the values of the quantum parameters. Explicitly, $g \in Z_h$ is a critical point of $\mathcal{F}_h$ precisely if $g\cdot B_- \in Y_B^*$ with $q_i(g\cdot B_-) = e^{\alpha_i(h)}$. On the other hand by [16] the quantum cohomology ring of the full flag variety $G^\vee/B^\vee$ is semisimple for generic choice of quantum parameters. This implies that $\mathcal{F}_h$ has precisely $\dim H^*(G^\vee/B^\vee) = |W|$ critical points, all non-degenerate, for generic h.

Following Givental [6], the critical points are directly related to integration cycles. Namely to any non-degenerate critical point p in Z_h, Givental associates a 'descending gradient cycle' for $\mathrm{Re}(\mathcal{F})$. In this way, one may obtain $|W|$ cycles in a generic mirror fiber Z_h, which should (conjecturally) provide a basis of the appropriate rapid decay homology group in the sense of Hien [8].

Definition 11.1. Consider the isomorphism $\delta_h : Z_h \to \mathcal{R}_{1,w_0}$ given by $g \mapsto g\cdot B_-$. Let $Z_h^{>0} := \delta_h^{-1}(\mathcal{R}_{1,w_0}^{>0})$.

Lemma 11.2. *Suppose $u\bar u^{-1} \in Z_0^{>0}$. Then $u \in U_+^{>0}$ and $\bar u \in U_-^{>0}$.*

Proof. It is clear from the definitions that $u \in U_+^{>0}$. The rest of the lemma follows from $u^{-1}\cdot B_- = \bar u^{-1}\cdot B_+$, together with Lusztig's result [18] that

$$U_+^{>0}\cdot B_- = U_-^{>0}\cdot B_+,$$

applied to the opposite pinning, where $\bar x_i(t) = \exp(-te_i)$ and $\bar y_i(t) = \exp(-tf_i)$. □

Proposition 11.3. *For any $h \in \mathfrak{h}_{\mathbb{R}}$ and $M \in \mathbb{R}_{>0}$, consider the set*

$$\mathcal{M}_{h,M} := \left\{ g \in Z_h^{>0} \mid \mathcal{F}(g;z) \leq \frac{1}{z} M \right\}.$$

If M is sufficiently large then $\mathcal{M}_{h,M}$ is a nonempty, compact subset of $Z_h^{>0}$. In particular the restriction of $\mathcal{F}_h$ to $Z_h^{>0}$ attains a minimum.

Note that by definition $\mathcal{M}_{h,M}$ is independent of the positive scalar z.

Proof. If $ue^h\bar{u}^{-1} \in \mathcal{M}_{h,M}$, then by Lemma 11.2 we have $u \in U_+^{>0}$ and $\bar{u} \in U_-^{>0}$. So we can fix a reduced expression $\mathbf{i}$ of w_0 and write

$$\begin{aligned} u &= x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_N}(a_N), \\ \bar{u} &= y_{i_1}(b_1)y_{i_2}(b_2)\cdots y_{i_N}(b_N), \end{aligned}$$

for positive a_i, b_i with $i \in I$. Using the a_i and b_i as coordinates for u and $\bar{u}$, respectively, we may define

$$\mathcal{N}_{h,M} := \{ ue^h\bar{u}^{-1} \in Z_h^{>0} \mid a_i \leq M,\ b_i \leq M \quad \forall i \in I \}.$$

Since $\mathcal{F}(ue^h\bar{u}^{-1};z) = \frac{1}{z}\left(\sum a_i + \sum b_i\right)$, it is clear that $\mathcal{M}_{h,M}$ is a closed subset of $\mathcal{N}_{h,M}$. It suffices therefore to show the following claim.

Claim: There exists an $m < M$ such that $\mathcal{N}_{h,M}$ is a subset of the compact set

$$\mathcal{N}_{h,M}^m := \{ ue^h\bar{u}^{-1} \in Z_h^{>0} \mid m \leq a_i \leq M \quad \forall i \in I \} \cong [m,M]^N.$$

Suppose indirectly that we have an index i and a sequence $u_{(s)}e^h\bar{u}_{(s)}^{-1}$ in $\mathcal{N}_{h,M}$ for which the coordinate $a_i \to 0$ as $s \to \infty$. We may assume that $u_{(s)}^{-1} \cdot B_-$ converges, passing to a subsequence if necessary. Then it follows that

$$\lim_{s\to\infty} (u_{(s)}^{-1} \cdot B_-) \in B_-\dot{w} \cdot B_-$$

for some $w < w_0$. On the other hand using $u_{(s)}^{-1} \cdot B_- = e^h\bar{u}_{(s)}^{-1} \cdot B_+$ we see that

$$\lim_{s\to\infty} (u_{(s)}^{-1} \cdot B_-) = e^h \lim_{s\to\infty} (\bar{u}_{(s)}^{-1} \cdot B_+) \in B_- \cdot B_+ = B_-\dot{w}_0 \cdot B_-.$$

Namely, this last limit cannot leave the big cell $B_- \cdot B_+$ because the b_i coordinates of the $\bar{u}_{(s)}$ are bounded from above by M.

Therefore we have arrived at a contradiction and the Claim is proved. $\square$

Corollary 11.4. *For every $h \in \mathfrak{h}_{\mathbb{R}}$ the function $\mathcal{F}_h$ has a totally positive critical point.*

The totally positive critical point is provided by the minimum of $\mathcal{F}_h$ on $Z_h^{>0}$. For type A this result was proved already in [26], where moreover it was shown that the totally positive critical point is unique, and this was used to describe the totally nonnegative part of the Peterson variety. One would hope to find uniqueness also in the general case, but this is still conjecture at the moment.

The same proof with the negative pinning also gives the following

Corollary 11.5. *For every $h \in \mathfrak{h}_{\mathbb{R}}$ the function $\mathcal{F}_h$ has a critical point in $\Gamma_h^{(w_0)}$.*

This critical point is given by a maximum of $\mathcal{F}_h$ on $\Gamma_h^{(w_0)}$. We might call it the totally negative critical point, and it is a special feature of the full flag variety that a symmetry between totally positive and totally negative critical points exists, compare [24, 26] and Lemma 5.3 in [28].

Assuming the conjecture about uniqueness of the totally positive/negative critical points in every fiber over $h \in \mathfrak{h}_{\mathbb{R}}$, it seems natural to think of the family $\Gamma^{(w_0)}$ of integration cycles as associated to the family of totally negative critical points, and the family $\Gamma^{(1)}$ of integration cycles as associated to the family of totally positive critical points (in both cases via Givental's construction). For SL_2 this is exactly the case, by direct calculation.

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